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THE CALCULATION OF ELECTROMAGNETIC FIELDS BY THE KIRCHHOFF-KOTTLER METHOD

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ABSTRACT

This document discusses some of the implications of the Kirchhoff-Kottler formulation for discontinuous surface distributions. The intrinsic impedance of free-space is recovered from the equations for the electric and magnetic fields, and the gain dependence of the antenna on frequency and geometry is identified. The frequency dependence of each integral is tabulated, and each is associated with a physical quantity on the reflecting surface. It is shown that the Kottler contour integral vanishes for closed surfaces, independent of field values on the boundary. The behavior of $\nabla\psi$, the gradient of the solution to the wave equation, is examined in a local context on the antenna surface to illustrate that charge distribution can give rise to transverse field components in the intermediate near-field region. Finally, derivations originally due to (1) J. A. Stratton and (2) M. I. Sancer are reviewed to bring out the fact that the Kirchhoff-Kottler formulation inherently satisfies the radiation conditions and vanishes the radial fields of scattering surfaces in the limit as the observer recedes to infinity. This important conclusion is established by two unique approaches. The development of Sancer also demonstrates that the Kottler contour integral can be evolved without any appeal to physical intuition, for the case of discontinuous illumination distributions, via the dyadic Green's function.

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GLOSSARY OF NOTATION

Symbol	Meaning
$\overline{E}(x', y', z')$, $\overline{H}(x', y', z')$	backscattered electric and magnetic fields
ψ, g	solutions to the wave equation
r, r′, ρ, ρ'	radius
k	wave number
$\overline{E}_1$, $\overline{H}_1$	fields of the illumination distribution
$\overline{n}$	unit normal vector to a surface
ds, $d\ell$	differential area and arc length
$\hat{i}$, $\hat{j}$, $\hat{k}$	Cartesian basis vectors
$\hat{1}_r$, $\hat{1}_\theta$, $\hat{1}_\phi$	spherical basis vectors
ω	angular frequency
μ, ϵ, σ	constitutive parameters: magnetic permeability, inductive capacity, electric conductivity
j, i	imaginary operator = $\sqrt{-1}$
$\overline{D}$, $\overline{B}$	dielectric displacement and magnetic flux density
Z_0	intrinsic impedance of free space
P_θ, P_ϕ	power contained in θ and ϕ polarized waves
O	order
I_i	integral
$\overline{J}$	current density
t	time

Symbol	Meaning
$\bar{\bar{I}}$	identity dyad
$\bar{\tau}$	tangent vector
$\bar{\bar{G}}$	free-space dyadic Green's function
L	operator $\nabla \times \nabla \times - k^2$
$\bar{V}$	arbitrary vector
$\propto$	proportionality symbol

THE CALCULATION OF ELECTROMAGNETIC FIELDS BY THE KIRCHHOFF-KOTTLER METHOD

INTRODUCTION

The starting-point of this document is the Kirchhoff-Kottler formulation for computing backscattered electric and magnetic fields from surfaces carrying charge and current distributions. Although standard textbooks[1] have carried this formulation for many years, the analytical difficulties associated with the evaluation of the integrals have delayed application to many practical antenna configurations. The advent of high-speed digital electronic computers has made possible the numerical evaluation of the otherwise intractable integrals, and has brought about renewed interest and a re-examination of the theory.

A few dates are interesting and significant. Maxwell's equations appeared around 1865, and Kirchhoff's theory was presented about 1882. By 1923 Kottler[2] had annexed his contour integral which, when taken in conjunction with Kirchhoff's results formed a theory which satisfied Maxwell's equations. More up-to-date discussion of the Kirchhoff-Kottler formulation appears in Physical Review by Stratton and Chu[3] in 1939, the textbook by Stratton[1] in 1941, and the well-known microwave antenna text by Silver[1] in 1949. Jones[4] mentions the Kottler integral again in the representation of the electromagnetic field in his 1964 text. Up to this time, it appears that the Kirchhoff integrals over current and charge distributions are regarded as basic, and the Kottler contour integral on boundary-line charge is considered a "correction" term or approximation. A 1964 textbook by Van Bladel,[5] which makes extensive use of the dyadic Green's function in the calculation of fields, comments on the linear charge density, and provides background for a unified approach to the diffraction formulation. In Radio Science, 1968, Sancer[6] presents a derivation of the general vector Kirchhoff equations, via the free-space dyadic Green's function, which intrinsically includes Kottler's boundary-line charge for open surfaces. The latter derivation is in a sense complete, and does not appeal to separate physical argument or intuition in achieving its objectives.

The Kirchhoff-Kottler formulation is usually presented in the notation of vector analysis, and constitutes a rather formidable array of symbols. It is abstract in

[1]Ref. 1, p. 469, Ref. 2, p. 160
[2]Ref. 3
[3]Ref. 4
[4]Ref. 5, p. 56
[5]Ref. 6, p. 221, p. 361
[6]Ref. 7

that the general formulation simply postulates existing fields $(\overline{E}_1, \overline{H}_1)$ which are to be integrated over some unspecified surface. The computer programmer usually notes that complex vectors are used throughout so that each of the eight original integrals decomposes into six integrals in the real domain (appropriately sorted according to their Cartesian vector components and the real-imaginary quadrature). Sometimes the program language and computer system allow direct vector operations and complex algebra. Nowhere, in the original formulation, is there an explicit connective between the sources and the illuminated surface.[1] The boundary conditions at the surface are not specified in the general formulation. In short, a theory descriptive of a physical process has been made available, not a flexible empirical formula which is adapted to a large class of problems on the basis of existing knowledge of antenna systems as is sometimes suggested. The theory is used exactly as it stands, without modification of any kind, and the test of its validity lies in comparison with other analyses and careful measurements on actual physical systems.

One way to remove the abstract character of the general formulation is to begin to specialize its application and examine the simplified result. Probably the most natural and practical assumption is that of a perfect conductor, leading to the concept of a sheet current and a charge distribution on the surface of the idealized conductor.[2] The assumption of an infinitely remote observer affords another opportunity to gain familiarity with the notation and the Kirchhoff-Kottler formulation. When these assumptions are made the gain dependence on frequency and geometry can be seen clearly for both $\overline{E}(x', y', z')$ and $\overline{H}(x', y', z')$. An identification of the radial and transverse spherical components of the composite field is easily made for the case of the distant observer.

Returning to the general formulation, and assuming a perfectly conducting surface, the intermediate near-field can be studied. The function of the term $\nabla\psi$, where $\psi = e^{-jkr}/r$ is taken as a solution to the wave equation, in a local context on the antenna surface can be related to an intuitive concept of fields due to point-charge distributions. Finally the question of whether or not the complete Kirchhoff-Kottler formulation satisfies the radiation condition[3] can be answered without resorting to numerical evaluation or reference to specific illuminations or reflector geometries. The vanishing of the Kottler integral for closed surfaces, under arbitrary illumination conditions, is obtained simultaneously while resolving the preceding question. It has been found that these exercises are invaluable in purging errors from a freshly written computer program and contribute to the overall appreciation of the Kirchhoff-Kottler diffraction theory. Most striking,

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[1] Ref. 2, p. 150, p. 418, are especially helpful in obtaining $\overline{E}_1$, $\overline{H}_1$
[2] Ref. 1, p. 37
[3] Ref. 1, p. 485, Ref. 2, p. 85

perhaps, is the internal self-consistency and close adherence to well-established physical principles that emerge through application of the theory.

GENERAL FORMULATION

The backscattered electric and magnetic fields of the general Kirchhoff-Kottler formulation are:

$$\bar{E}(x', y', z') = -\frac{1}{j\omega\epsilon}\,\frac{1}{4\pi}\oint_c \nabla\psi\,\bar{H}_1 \cdot d\bar{\ell}$$

① (Kottler)

$$-\frac{1}{4\pi}\int_{s_1}\left[j\omega\mu\left(\bar{n}\times\bar{H}_1\right)\psi + \left(\bar{n}\times\bar{E}_1\right)\times\nabla\psi + \left(\bar{n}\cdot\bar{E}_1\right)\nabla\psi\right]ds$$

② ③ ④

(Kirchhoff)

$$\bar{H}(x', y', z') = \frac{1}{j\omega\mu}\,\frac{1}{4\pi}\oint_c \nabla\psi\,\bar{E}_1 \cdot d\bar{\ell}$$

⑤ (Kottler)

$$+\frac{1}{4\pi}\int_{s_1}\left[j\omega\epsilon\left(\bar{n}\times\bar{E}_1\right)\psi - \left(\bar{n}\times\bar{H}_1\right)\times\nabla\psi - \left(\bar{n}\cdot\bar{H}_1\right)\nabla\psi\right]ds$$

⑥ ⑦ ⑧

(Kirchhoff)

where

$$\psi = \frac{e^{-jkr}}{r}, \qquad \nabla\psi = -\left(jk + \frac{1}{r}\right)\psi\,\hat{1}_r.^{1}$$

In the above, $\bar{E}_1$ and $\bar{H}_1$ are electric and magnetic distributions which exist in space and may or may not be associated with an actual physical surface. When a material surface is chosen, appropriate boundary conditions are applied to evaluate $\bar{E}_1$, $\bar{H}_1$ in terms of the incident fields $\bar{E}_i$, $\bar{H}_i$. Vectors such as $\bar{n}$, $\bar{E}_1$, $\bar{H}_1$, $d\bar{\ell}$, and $\nabla\psi$ are ordinarily specified by their Cartesian components and associated with a definite point of application in the problem. They are bound,[2] not to the origin, but to the surface and are taken in a local context. These vectors are part of an integration on the four-dimensional manifold $\left(\hat{1}_x, \hat{1}_y, \hat{1}_z, j\right)$ where the Cartesian basis vectors are fixed in space. The resulting vector fields $\bar{E}(x', y', z')$, $\bar{H}(x', y', z')$ are usually rewritten in terms of projections on the moving spherical triad $\left(\hat{1}_r, \hat{1}_\theta, \hat{1}_\phi\right)$ and a phase angle.

SPECIALIZATION OF THE GENERAL FORMULATION

When the general formulation is applied to the class of problems for which the illumination distributions reside on perfectly conducting surfaces, the Kirchhoff-Kottler theory reduces to:[3]

$$\bar{E}(x', y', z') = -\frac{1}{j\omega\epsilon}\frac{1}{4\pi}\oint_c \underbrace{\nabla\psi\bar{H}_1 \cdot d\bar{\ell}}_{\text{(1) radial}} - \frac{1}{4\pi}\int_{s_1}\left[\underbrace{j\omega\mu\left(\bar{n}\times\bar{H}_1\right)\psi}_{\text{(2) transverse \& radial}} + \underbrace{\left(\bar{n}\cdot\bar{E}_1\right)\nabla\psi}_{\text{(4) radial}}\right] ds$$

$$\bar{H}(x', y', z') = -\frac{1}{4\pi}\int_{s_1}\underbrace{\left(\bar{n}\times\bar{H}_1\right) x\nabla\psi\, ds}_{\text{(7) transverse}}$$

[1]Silver's conventions are used for ψ in the above.

[2]Ref. 8, pp. 33-34, Ref. 9, pp. 5-6. These references distinguish several kinds of vectors.

[3]The component fields are radial and transverse, as indicated, for an infinitely remote observer.

where

$$\psi = \frac{e^{-jkr}}{r}, \qquad \nabla\psi = -\left(jk + \frac{1}{r}\right)\psi\,\hat{1}_r \text{ [1]}$$

as before.

The simplification is due to the boundary conditions[1]

$$\left.\begin{array}{llll} \bar{n}\cdot(\bar{B}_2-\bar{B}_1) = 0 & & \bar{n}\times(\bar{E}_2-\bar{E}_1) = 0 & \\ \bar{n}\cdot(\bar{D}_2-\bar{D}_1) = \eta & & \bar{n}\times(\bar{H}_2-\bar{H}_1) = \bar{K}_2 & \end{array}\right| \begin{array}{l}\sigma_1 \to 0 \\ \sigma_2 \to \infty\end{array}$$

$$B_2 = \bar{H}_2 = \bar{D}_2 = \bar{E}_2 = 0 .$$

Physically, the antenna surface is considered locally plane, and the normal components of $\bar{H}_1$ and the tangential components of $\bar{E}_1$ vanish at the interface. In effect, the normal $\bar{E}_1$ and tangential $\bar{H}_1$ fields are converted into electric charge distributions, and electric sheet currents flowing along the local tangent to the surface. The contour integral can be regarded as a discontinuity of the surface sheet current resulting in a surfeit or deficit of electric charge.

The vector mechanics by which the Kirchhoff-Kottler formulation achieves results satisfying physical intuition about the problem may be worth reviewing. Vector dot- and cross-products can serve as mnemonic devices for identifying sheet currents lying in the reflecting surface, surface charges, and boundary charges. It is fundamental that a dot-product amounts to a projection between two vectors and a cross-product always results in a vector which is orthogonal to the given pair of vectors. From this it follows that $(\bar{n}\cdot\bar{E}_1)\,\nabla\psi$ of integral 4 is due to that part of $\bar{E}_1$ which is orthogonal to the surface and induces charge. Since $(\bar{n}\cdot\bar{E}_1)\,\nabla\psi$ is radial, this component of the total field acts radially on an

[1]Ref. 1, pp. 34-37

observer, as do all fields from charges.[1] Also the term $(\bar{n} \times \bar{H}_1)$ must be a vector which is orthogonal to both $\bar{n}$ and $\bar{H}_1$. If $\bar{n} \times H_1$ is orthogonal to $\bar{n}$ it must be in the tangent plane of the surface, and is suited for describing a sheet current due to tangential $\bar{H}_1$ as in integral ②. Depending on the orientation of $\bar{n}$ over the surface, $(\bar{n} \times \bar{H}_1)$ can obviously exhibit either radial or transverse fields at the point of the observer. In integral ⑦ the sheet current is crossed, vectorially, into $\nabla\psi$, which is purely radial. Then the integrand $(\bar{n} \times \bar{H}_1) \times \nabla\psi$ is, necessarily purely transverse, as is intuitively required of the scattered magnetic field. Finally, the contribution of integral ① is radial since $\bar{H}_1 \cdot d\bar{\ell}$ is a scalar and $\nabla\psi$ is purely radial. Since $d\bar{\ell}$ lies in the surface along the contour and is a tangent to it in a local context, the magnitude of $\bar{H}_1 \cdot d\bar{\ell}$ depends on the relative aspect between $\bar{H}_1$ and $d\bar{\ell}$ according to the projection rule. Obviously the tangential part of $\bar{H}_1$ affects the magnitude of the charge on the contour as stated previously.

The preceding conclusious about the radial and transverse nature of the field components were obtained on the assumption that the observer was infinitely remote and the surface was perfectly conducting. It could be seen that $\bar{H}(x', y', z')$ was transverse since any vector crossed into a radial $\nabla\psi$ is transverse, and only a single integral provided the solution for the scattered magnetic field. The scattered electric field $\bar{E}(x', y', z')$ contained one integral with both transverse and radial components, and two other integrals were shown to contribute only radial fields. It is not a priori evident that the three radial components combine cooperatively, in general, in the Kirchhoff-Kottler theory and annihilate to satisfy the radiation condition.[2] On the assumption that the radial fields might vanish, the remaining transverse fields should yield a well-known ratio, the intrinsic impedance of free space.

$$\frac{|\bar{E}(x', y', z')|}{|\bar{H}(x', y', z')|} = \frac{\left| -\frac{1}{4\pi} \int_{s_1} j\omega\mu(\bar{n} \times \bar{H}_1)\ \psi\, ds \right|}{\left| -\frac{1}{4\pi} \int_{s_1} (\bar{n} \times \bar{H}) \times \nabla\psi\ ds \right|} = \frac{\left| j\omega\mu \int_{s_1} (\bar{n} \times \bar{H}_1)\ \psi\, ds \right|}{\left| -jk \int_{s_1} (\bar{n} \times \bar{H}_1) \times \hat{1}_r\ \psi\, ds \right|}$$

$$= \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}} = Z_0 \qquad \text{(free space)}$$

[1] Ref. 1, Appendix I, p. 602 gives the dimensions of the physical quantities that also assist in identifying the behavior of the integrands. The physical constants ω, μ, and ϵ are not included in the discussion above, and only the vector character of the integrands is examined.

[2] The discussion of the proof is deferred to the latter sections of this document. See Appendix A and Appendix B.

The cross-product involving $\hat{1}_r$ in the denominator causes no difficulty here. Since it has been assumed, tentatively, that the fields are purely transverse for the infinitely remote observer, it follows that

$$\left| \int_{s_1} \left(\bar{n} \times \bar{H}_1 \right) \times \hat{1}_r \, ds \right| = \left| jk\hat{1}_r \times \int_{s_1} \left(\bar{n} \times \bar{H}_1 \right) \psi \, ds \right|$$

$$= \left| jk\hat{1}_r \right| \left| \int_{s_1} \left(\bar{n} \times \bar{H}_1 \right) \psi \, ds \right| \sin \underline{/\hat{1}_r, \bar{E}(x', y', z')}$$

$$= k \left| \int_{s_1} \left(\bar{n} \times \bar{H}_1 \right) \psi \, ds \right|$$

From this preliminary analysis it appears that the Kirchhoff-Kottler theory can provide a diverging spherical backscattered wave, subject to additional analysis concerning the vanishing of the radial fields.

Additional confidence in the description of the backscattered fields can be obtained by postulating a pair of incident fields $\bar{E}_i$, $\bar{H}_i$ which give a Poynting vector $\bar{P}_i = \bar{E}_i \times \bar{H}_i$ that is directed toward some scatterer. A locally plane segment of the scatterer can be taken, and the directions of the scattered fields are assigned by the integrals (2) and (7) of the theory, with due regard for signs. It is easy to verify that the scattered fields give a Poynting vector $\bar{P}_s = \bar{E}(x', y', z') \times \bar{H}(x', y', z')$ away from the reflector. This result, although somewhat trivial in appearance, is important since subsequent applications utilize the backscattered fields to illuminate other surfaces (i.e. dual reflector systems such as those of Cassegrain and Gregory).[1] The $\bar{E}(x', y', z')$, $\bar{H}(x', y', z')$ comprise a new illumination function, effectively, and determine a new $\bar{E}_1$, $\bar{H}_1$, but the incident fields are not, in general, related simply to one another as in the case of a spherical wave.

[1]Ref. 1, pp. 131-137

A subject of considerable interest in the antenna field is directive gain, which is defined as[1]

$$G(\theta, \phi) = \frac{P(\theta, \phi)}{\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} P(\theta, \phi) \sin\theta \, d\theta \, d\phi}$$

$$= \frac{P_\theta (\theta, \phi) + P_\phi (\theta, \phi)}{\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} P_\theta (\theta, \phi) \, ds + \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} P_\phi (\theta, \phi) \, ds}$$

This topic will not be discussed at length but it is interesting to discover how gain variations are introduced in the Kirchhoff-Kottler formulation.

A special class of scatterers is considered, initially, for which all the differential contributions of integrals 2 or 7 arrive in-phase at the point of the observer. Focal-point fed paraboloids, observed on-axis, are familiar examples. The directive gain of such reflectors is usually given by

$$G(\theta, \phi) = c\left(\frac{\pi d}{\lambda}\right)^2$$

where c is an efficiency factor related to illumination edge taper. For a fixed geometry diameter d is fixed and

$$G(\theta, \phi) \propto \omega^2$$

In the Kirchhoff-Kottler formulation, considering both the electric and magnetic fields, the same frequency dependence is required by intuition.

[1]Ref. 9, pp. 179-180, Ref. 2, p. 90

Since a cophased aperture exists by assumption,

$$G(\theta, \phi) \propto |\overline{E}(x', y', z')|^2 \propto \omega^2$$

by integral 2. Similarly, the identical gain must come about for the magnetic field, integral 7, through $\nabla\psi$ since no coefficients appear.

$$G(\theta, \phi) \propto |\overline{H}(x', y', z')|^2 \propto |\nabla\psi|^2 = \left| -\left(jk + \frac{1}{r}\right)\psi \hat{1}_r \right|^2$$

In the far-field, only $O(1/r)^1$ terms remain, and the wave number $k = 2\pi/\lambda$ establishes the fact that

$$G(\theta, \phi) \propto |\overline{H}(x', y', z')|^2 \propto \omega^2$$

It is anticipated, therefore, that

$$\int_{s_1} \left(\overline{n} \times \overline{H}_1\right) \psi \, dS$$

is invariant with frequency for cophased apertures of the type assumed, observed on-axis, providing a check on computer programs. Useful decisions, concerning the need for double precision, sampling etc., can be made by insisting on this ω invariance of the integral.

When the cophased aperture assumption is removed, it can be seen that

$$\int_{s_1} (\bar{n} \times \bar{H}_1)\, \psi\, dS$$

leads to complex phasor polygons which are now not straight lines, but may have contributions tending to annihilate to some extent. They yield reduced field strengths, and imply gain degradation of the system. In the far-field region, each transverse component E_θ, E_ϕ of the scattered field leads to a phasor polygon. No attempt will be made here to relate the envelope or shape of the scattered radiation pattern to absolute signal strength consistent with energy conservation principles via the integral, over all space, as required by the definition of gain. The implications of the superposition principle (a linear concept) and the notion of energy conservation (a square-law for fields) are not peculiar to the Kirchhoff-Kottler formulation.[1]

Having discussed the frequency dependence of integrals ② and ⑦ in the far-field for a cophased aperture system, integrals ① and ④ are examined as these complete the formulation under the assumption of perfect conductivity. Since $\nabla\psi$ implies the first power of ω, it follows that the Kottler contour integral is frequency <u>independent</u> due to the presence of the coefficient ω^{-1} outside of the integral. The surface charge integral ④, however is without any coefficient and varies directly as the first power of ω or frequency. To sum up:

$$\bar{E}(x', y', z') = \underset{\text{invariant}}{I_{①}(\omega^{\circ})} + I_{②}(\omega) + I_{④}(\omega)$$

$$\bar{H}(x', y', z') = I_{④}(\omega)$$

for simple cophased apertures observed on-axis.

THE INTERMEDIATE NEAR-FIELD

Returning to the general formulation, and assuming a perfectly conducting surface, only the integrals ①, ②, and ④ of the electric field and the integral ⑦

[1]Ref. 10, pp, 101-105.

of the magnetic field remain and are examined here in some detail. The behavior of the sum of these integrals in the intermediate near-field region is of interest, as is the special case when the observer recedes to infinity. Also, prior to becoming engrossed in computational aspects of the problem several questions might be asked concerning the Kirchhoff-Kottler formulation:

(1) Does the solution for the scattered field satisfy Maxwell's equations?

(2) Can one expect the solution to satisfy the radiation conditions at infinity? (i.e. Is there a net radial field component?)

(3) Is the Kottler integral an appended "correction" term (approximation) based on physical intuition or argument?

(4) What is the behaivor of the Kottler integral for closed surfaces since the contour C was originally for open surfaces?

The answers to these questions can be obtained without resorting to evaluation involving particular geometries, illumination, etc., even though the task is tedious and involves numerous manipulations on the field expressions.

For the reader who is interested primarily in the application of the theory, and to a lesser degree in the mathematical arguments underlying it, the answers are tabulated here. The derivations can be found in the Appendix of this document.

(1) The Kirchhoff-Kottler formulation leads to solutions $\overline{E}(x', y', z')$, $\overline{H}(x', y', z')$ which satisfy Maxwell's equations.

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}, \qquad \nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \qquad \text{(free-space)}$$

(2) The radiation conditions are satisfied by the Kirchhoff-Kottler formulation and the radial components of integrals 1 , 2, and 4 annihilate for the remote observer of the electric field. The magnetic field is inherently transverse at infinity since the radial field is precluded by a cross-product involving a radial vector.

$$\lim_{r \to \infty} r\overline{E}(x', y', z') \text{ is finite,}$$

$$\lim_{r\to\infty} r\left[\hat{1}_r \times \overline{H}(x', y', z') + \left(\frac{\epsilon}{\mu}\right)^{1/2} \overline{E}(x', y', z')\right] = 0$$

$$\lim_{r\to\infty} r\overline{H}(x', y', z') \text{ is finite,}$$

$$\lim_{r\to\infty} r\left[\left(\frac{\epsilon}{\mu}\right)^{1/2} \hat{1}_r \times \overline{E}(x', y', z') - \overline{H}(x', y', z')\right] = 0$$

The proof lies in a development[1] which shows that both $\overline{E}(x', y', z')$ and $\overline{H}(x', y', z')$ are divergenceless, so that

$$\nabla' \cdot \overline{E}(x', y', z') = \nabla' \cdot \overline{H}(x', y', z') = 0$$

Further evidence is presented by a separate proof due to Sancer[2] who **shows that**

$$\overline{E}(x', y', z') = \frac{j\omega\mu}{4\pi}\, \psi\left(\overline{\overline{I}} - \hat{1}_r\, \hat{1}_r\right) \cdot \int_{s_1} \left(\overline{n} \times \overline{H}_1\right) e^{\,jk\overline{\rho}\cdot\hat{1}_r}\, ds \qquad \text{as } r \to \infty$$

$$\overline{H}(x', y', z') = jk\psi\, \hat{1}_r \times \int_{s_1} \left(\overline{n} \times \overline{H}_1\right) e^{\,jk\overline{\rho}\cdot\hat{1}_r}\, ds \qquad \text{as } r \to \infty .$$

The latter development substantiates in detail an assertion by Silver[3] that the far-zone field is transverse by the current distribution method (i.e. by the Kirchhoff-Kottler method).

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[1]Ref. 1, pp. 469-470; and Appendix A, this document
[2]Ref. 7, p. 143; and Appendix B, this document
[3]Ref. 2, p. 149

(3) The Kottler integral, which is introduced via a physical argument that envisions a discontinuity in a sheet current is shown by Sancer to evolve naturally using the free-space dyadic Green's function.

$$\bar{\bar{G}} = \left(\bar{\bar{I}} + \frac{1}{k^2} \nabla \nabla g\right)$$

where

$$g = \frac{\psi}{4\pi} = \frac{1}{4\pi} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r} - \bar{r}'|} .$$

It appears that the Kottler contour term is more than a "correction" term that incidentally leads to the satisfaction of the Maxwell equations.

(4) The Kottler integral for a <u>closed</u> surface vanishes identically for arbitrary illumination distribution because a dyad analogue to Stokes' theorem[1] is applicable and clearly shows that the contour itself vanishes. That is

$$\oint_{s_1} \bar{n} \cdot \nabla x(\bar{H}_1 \nabla\psi)\, dS = \oint_{c_1} d\bar{\ell}_1 \cdot (\bar{H}_1 \nabla\psi) + \oint_{c_2} d\bar{\ell}_2 \cdot (\bar{H}_1 \nabla\psi)$$

$$= \oint_c \nabla\psi \bar{H}_1 \cdot d\bar{\ell} = 0$$

[1]See Appendix A for the vector and dyadic form of the curl theorem of Stokes on an open surface.

since

$$d\bar{\ell}_1 = - d\bar{\ell}_2 .$$

The intermediate near-field formulation is now examined. As shown in Fig. 1, an arbitrary reflecting surface, feed-point, and point of observation are considered. The phase and magnitude of the wave front arriving at the surface is affected by a factor of the form $e^{-jk\rho'}/\rho'$, and accounts for the path length traversed in passing from feed to reflector. This is already stated implicitly in the theory when $\bar{E}_1$ and $\bar{H}_1$ are specified at γ_1.

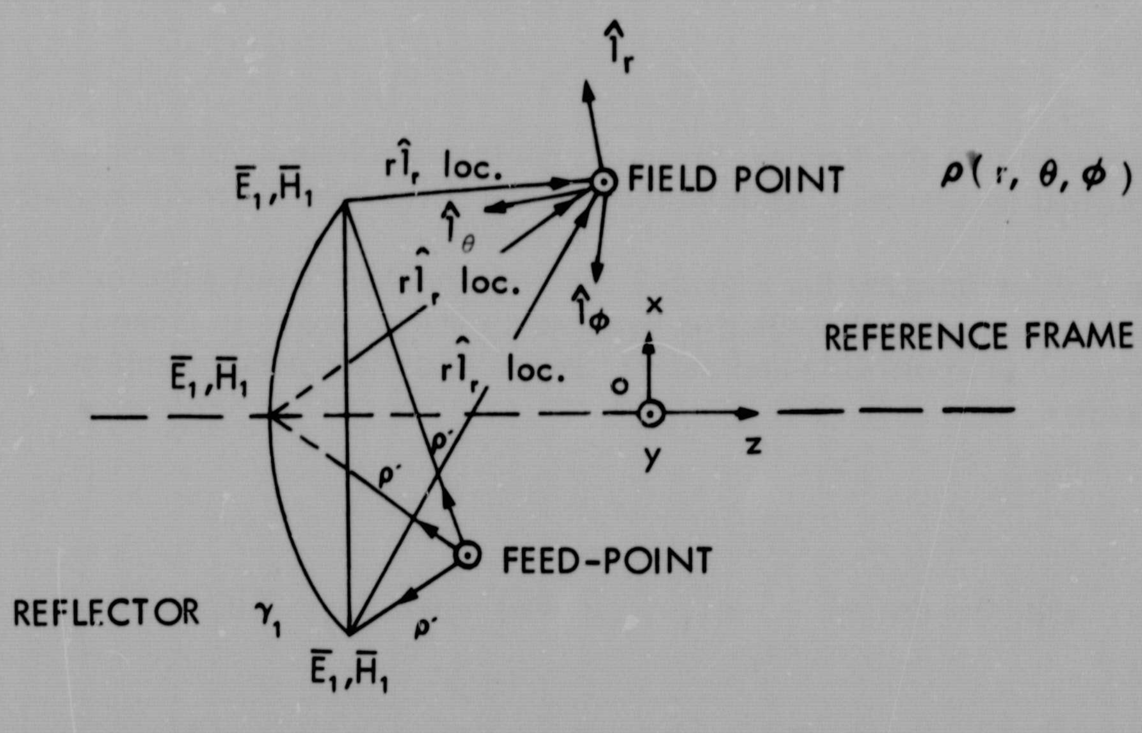

Figure 1. The Intermediate Near-Field Region

Only three paths are traced from the feed to the observer to suggest the process being described. It can be seen that the distance and direction from feed to surface must be assumed unique, and the same is true of the distance and direction from surface to observer. A factor e^{-jkr}/r is evidently required for the latter part of the path. Over the entire path the product $(e^{-jk\rho}/\rho)(e^{-jkr}/r)$ describes the amplitude and phase of a propagated wave.

In the Kirchhoff-Kottler formulation the history of the fields emanating from the surface is of concern, and is given by the quantities $\psi = e^{-jkr}/r$ and $\nabla\psi$

$= -(jk + 1/r)\psi \hat{1}_r$. These factors are under the integral sign, in general, since each differential area is uniquely affected. Since the charge distributions are not associated with propagation, but fields, $\nabla\psi$ appears instead of ψ above. By this device, the formulation introduces terms which descend as $O(1/r)^1$ and $O(1/r)^2$ and simultaneously achieves the (local) radial character of those fields arising from charge distribution. See integrals ① and ④ . Only the quantity ψ appears for integral ②, and the orientation of this field component is determined by the direction of a (local) unit tangent, $(\bar{n} \times \bar{H}_1)/|\bar{n} \times \bar{H}_1| = \bar{\tau}$. The (total) field given by integral ⑦ is purely transverse (in a local context) since, effectively, the tangent $\bar{\tau}$ is crossed into a radial vector $\hat{1}_r$ contained in $\nabla\psi$.

To conclude, it is noted that the backscattered fields of the Kirchhoff-Kottler formulations are formed in Cartesian components E_x, E_y, E_z and later resolved into spherical components E_r, E_θ, E_ϕ. From Figure 1 is can be seen that supposed radial fields, which can only attract or repel along a line of action between a differential area ds and the observer, can have an E_θ and/or E_ϕ resolute with respect to the moving spherical triad of basis vectors. Likewise, supposed transverse fields arising from a differential area in such a manner that they are orthogonal to $\nabla\psi$ (or $\hat{1}_r$), can have radial resolutes relative to the spherical system. The $\hat{1}_r$, $\hat{1}_\theta$, $\hat{1}_\phi$ basis vectors are shown in Figure 1 to illustrate this fact. The notion that the $\hat{1}_\theta$, $\hat{1}_\phi$ vectors lead to a description of a transverse field is not particularly useful, therefore, in the intermediate near-field region. Mathematically the resolution into spherical components is correct but it does not appear to aid the understanding of the physical aspects of the problem. Also, since fields other than those synchronous and orthogonal fields associated with a spherical wave are now obtained in the solution, it appears that dual reflector systems may be measurably affected by induction as well as propagation terms.

SUMMARY

This document assumed the Kirchhoff-Kottler diffraction formulation presented in the literature and discussed some of the implications of theory for different specializations. A consideration of the behavior of the formulation when the reflectors are idealized open or closed surfaces, and when the observer is at a large distance from the scattering object, for example, provided valuable connectives between the mathematical theory and the physics of the problem. Recovery of the intrinsic impedance of free space from the ratio of $\bar{E}(x', y', z')$ to $\bar{H}(x', y', z')$, and the gain dependence on ω^2, as well as the vanishing of the radial fields at infinity, not only satisfied intuitive ideas on diffraction, but also provided means for verifying a computer program. Satisfaction of Maxwell's equations by the general solutions for the scattered fields was also demonstrated. Finally, the derivation of the Kirchhoff-Kottler formulation in a direct manner via the dyadic Green's function afforded an interesting departure from the historical development.

In conclusion, the objective of this document was to provide an overview of the Kirchhoff-Kottler formulation and stimulate further interest in diffraction studies by their method of analysis coupled with present-day computer technology. The modular Fortran diffraction program developed at Goddard Space Flight Center by the Antenna Systems Branch has been used in a large variety of applications including monopulse configurations, thermally distorted satellite antennas, large arrays of source elements and dual-reflector systems. While the mathematics and theoretical discussions may at times appear abstract, the ultimate objective has been that of determining the <u>extent</u> of the inherent soundness of the Kirchhoff-Kottler approach for simulating objects in the physical world. The combined effect of wave curvature, surface curvature and size, and observer range and angle on this 88-year old theory are still not generally known.

"...to pursue mathematical analysis while at the same time turning one's back on its applications and on intuition is to condemn it to hopeless atrophy." (R. Courant)[1] Increased computer capability should enhance applications and thereby clarify the range of validity of the theory.

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APPENDIX A

STRATTON'S PROOFS (FIELD DIVERGENCE AND MAXWELL'S EQUATIONS)[1]

Stratton has shown that the Kirchhoff-Kottler formulation for $\overline{E}(x', y', z')$, consisting of four integrals, has zero divergence and leads to waves which are transverse at great distances from the sources. He has also shown that $\nabla' \times \overline{H}(x', y', z') = (\partial/\partial t)\,\overline{E}(x', y', z')$. It is stated that $\overline{H}(x', y', z')$ has zero divergence, and $\nabla' \times \overline{E}(x', y', z') = -(\partial\overline{B}/\partial t)(x', y', z')$. The former proofs are carried out on the general formulation, and the conclusions are therefore, also valid when the boundary conditions for a perfect reflector $(\sigma \to \infty)$ are invoked.

In this appendix the proofs of Stratton are given in detail for both $\nabla' \cdot \overline{E}(x', y', z')$ and $\nabla' \cdot \overline{H}(x', y', z')$. For completeness, $\nabla' \times \overline{E}(x', y', z') = -(\partial/\partial t)\,\overline{B}(x', y', z')$ and $\nabla' \times \overline{H}(x', y', z') = (\partial/\partial t)\,\overline{E}(x', y', z')$ are both verified. Stratton's operator $\nabla_{\nabla\psi}$ is not used, and associations such as $\nabla \cdot \left(\overline{H}_1\, \nabla\psi\right)$ are treated as the divergence of a dyad $\left(\overline{H}_1\, \nabla\psi\right)$, since neither a vector dot- or cross-product is indicated. All of the terms arising through the use of vector identities are written out in detail. The arguments which subsequently vanish some of these terms are then presented. In some instances identities could not be found for operators, vectors, and dyads and only identities for vectors and dyads were available. In such instances detailed expansions were actually written as indicated, but will not be given here as they are easily verified. It was not assumed that an operator could be substituted for a vector in these cases. Some of the more tedious expansions involved as many as 72 terms to prove an identity relationship, and these are not reproduced here. Even so, the derivations still tend to be tedious; however, they provide valuable insight concerning the behavior of the Kirchhoff-Kottler formulation.

TABULATION OF IDENTITIES

$$\nabla \cdot (\phi\overline{A}) \equiv \nabla\phi \cdot \overline{A} + \phi\nabla \cdot \overline{A} \tag{I}$$

$$\nabla \cdot (\overline{A} \times \overline{B}) \equiv \overline{B} \cdot \nabla \times \overline{A} - \overline{A} \cdot \nabla \times \overline{B} \tag{II}$$

[1]Ref. 1, pp. 469-470.

$$\nabla \times (\nabla \phi) \equiv 0 \tag{III}$$

$$\int_s (\nabla \times \overline{A}) \cdot \overline{n}\, dS = \oint_c \overline{A} \cdot d\overline{\ell} \qquad \text{curl theorem of Stokes} \tag{IV}$$

$$\nabla \times (\phi \overline{A}) = \phi \nabla \times \overline{A} + \nabla \phi \times \overline{A} \tag{V}$$

$$(\overline{A} \times \overline{B}) \cdot \overline{C} = \overline{A} \cdot (\overline{B} \times \overline{C}) \tag{VI}$$

$$\nabla \times (\overline{A} \times \overline{B}) = (\overline{A}\nabla \cdot - \overline{A} \cdot \nabla)\,\overline{B} - (\overline{B}\nabla \cdot - \overline{B} \cdot \nabla)\,\overline{A} \tag{VII}$$

$$\oint_c d\overline{\ell} \cdot \overline{\overline{D}} = \int_s \overline{n} \cdot \nabla \times \overline{\overline{D}}\, ds \qquad \text{dyad analogue of Stokes' theorem} \tag{VIII}$$

$$\overline{A} \cdot (\overline{B} \times \overline{\overline{D}}) = (\overline{A} \times \overline{B}) \cdot \overline{\overline{D}} \qquad \text{where } \overline{\overline{D}} \text{ is a dyad} \tag{IX}$$

$$\int_v \nabla \cdot \overline{A}\, dv = \oint_s \overline{n} \cdot \overline{A}\, ds \qquad \text{divergence theorem of Gauss} \tag{X}$$

$$\int_v \nabla \cdot \overline{\overline{D}}\, dv = \oint_s \overline{n} \cdot \overline{\overline{D}}\, ds \qquad \text{dyad analogue of Gauss' theorem} \tag{XI}$$

THE TRANSVERSE ELECTRIC FIELD

Repeating the general formulation for the scattered electric field:

$$\bar{E}(x', y', z') = -\frac{1}{j\omega\epsilon}\frac{1}{4\pi}\oint_c \nabla\psi\bar{H}_1 \cdot d\bar{\ell}$$

$$-\frac{1}{4\pi}\int_{s_1}\left[j\omega\mu(\bar{n}\times\bar{H}_1)\ \psi + (\bar{n}\times\bar{E}_1)\times\nabla\psi + (\bar{n}\cdot\bar{E}_1)\ \nabla\psi\right] ds$$

The transverse character of the field at great distances follows from[1]

$$\nabla' \cdot \bar{E}(x', y', z') = -\frac{1}{j\omega\epsilon}\frac{1}{4\pi}\oint_c \nabla'(\bar{H}_1 \cdot d\bar{\ell}) \cdot \nabla\psi + (\bar{H}_1 \cdot d\bar{\ell})\nabla' \cdot (\nabla\psi) \quad \text{(I)}$$

$$-\frac{1}{4\pi}\int_{s_1} j\omega\mu\left[\nabla'\psi \cdot (\bar{n}\times\bar{H}_1) + \psi\nabla' \cdot (\bar{n}\times\bar{H}_1)\right] ds \quad \text{(I)}$$

$$-\frac{1}{4\pi}\int_{s_1}\left[\nabla\psi \cdot \nabla'\times(\bar{n}\times\bar{E}_1) - (\bar{n}\times\bar{E}_1) \cdot \nabla'\times(\nabla\psi)\right] ds \quad \text{(II), (III)}$$

$$-\frac{1}{4\pi}\int_{s_1}\left[\nabla'(\bar{n}\cdot\bar{E}_1) \cdot (\nabla\psi) + (\bar{n}\cdot\bar{E}_1)\ \nabla' \cdot (\nabla\psi)\right] ds \quad \text{(I)}$$

[1]The Roman numeral designates the tabulated identity used.

Since $\nabla' = -\nabla$ when the operators apply to ψ or its derivatives;[1,2] and since the illumination distribution is not affected by a change of coordinates on the part of the observer (x', y', z'),

$$\nabla' \cdot \overline{E}(x', y', z') = \frac{1}{j\omega\epsilon}\frac{1}{4\pi}\oint_c \nabla^2 \psi \overline{H}_1 \cdot d\overline{\ell}$$

$$+ \frac{1}{4\pi}\int_{s_1} \left[j\omega\mu\left(\overline{n} \times \overline{H}_1\right) \cdot \nabla\psi + \left(\overline{n} \cdot \overline{E}_1\right) \nabla^2 \psi\right] ds$$

$$= \frac{-k^2}{j\omega\epsilon}\frac{1}{4\pi}\oint_c \psi \overline{H}_1 \cdot d\overline{\ell} - \frac{k^2}{4\pi}\int_{s_1} \left(\overline{n} \cdot \overline{E}_1\right) \psi \, ds$$

$$+ \frac{j\omega\mu}{4\pi}\int_{s_1} \left(\overline{n} \times \overline{H}_1\right) \cdot \nabla\psi \, ds$$

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[1] Ref. 1, p. 469, p. 169.

[2] Stratton uses $\phi = e^{+jkr}/r$ and Silver uses $\psi = e^{-jkr}/r$ for the solution to the homogeneous wave equation $\nabla^2 \psi + k^2 \psi = 0$. In this document ϕ is reserved for a spherical coordinate or a generic scaler with identities. Take $\psi = e^{+jkr}/r$ above.

$$\int_{s_1} \left[\nabla \times \left(\psi \overline{H}_1\right)\right] \cdot \overline{n}\, ds = \oint_c \psi \overline{H}_1 \cdot d\overline{\ell}$$

$$= \int_{s_1} \left(\psi \nabla \times \overline{H}_1 + \nabla \dot{\psi} \times \overline{H}_1\right) \cdot \overline{n}\, ds \qquad \textbf{(IV), (V)}$$

$$-\left(\nabla \psi \times \overline{H}_1\right) \cdot \overline{n} = \left(\overline{n} \times \overline{H}_1\right) \cdot \nabla \psi \qquad \textbf{(VI)}$$

$$\nabla' \cdot \overline{E}(x', y', z') = -\frac{k^2}{j\omega\epsilon}\frac{1}{4\pi}\oint_c \psi \overline{H}_1 \cdot d\overline{\ell} - \frac{k^2}{4\pi}\int \left(\overline{n} \cdot \overline{E}_1\right) \psi\, ds$$

$$-\frac{j\omega\mu}{4\pi}\oint_c \psi \overline{H}_1 \cdot d\overline{\ell} + \frac{j\omega\mu}{4\pi}\int_{s_1} \psi \nabla \times \overline{H}_1 \cdot \overline{n}\, ds$$

Then

$$\nabla' \cdot \overline{E}(x', y', z') = 0$$

since

$$\nabla \times \overline{H}_1 = \overline{J} + \frac{\partial \overline{D}}{\partial t} = \epsilon \frac{\partial}{\partial t}\left(\overline{E}_1 e^{-j\omega t}\right) = -j\omega\epsilon\, \overline{E}_1 e^{-j\omega t} .$$

THE TRANSVERSE MAGNETIC FIELD

Repeating the general formulation for the scattered magnetic field:

$$\bar{H}(x', y', z') = \frac{1}{j\omega\mu}\,\frac{1}{4\pi}\oint_c \nabla\psi\,\bar{E}_1 \cdot d\bar{\ell}$$

$$+ \frac{1}{4\pi}\int_{s_1}\left[j\omega\epsilon(\bar{n}\times\bar{E}_1)\,\psi - (\bar{n}\times\bar{H}_1)\times\nabla\psi - (\bar{n}\cdot\bar{H}_1)\,\nabla\psi\right] ds$$

The transverse character of the field at great distances follows from

$$\nabla' \cdot \bar{H}(x', y', z') = \frac{1}{j\omega\mu}\,\frac{1}{4\pi}\oint_c \nabla'(\bar{E}_1 \cdot d\bar{\ell})\cdot\nabla\psi + (\bar{E}_1 \cdot d\bar{\ell})\,\nabla'\cdot\nabla\psi \qquad \text{(I)}$$

$$+ \frac{1}{4\pi}\int_{s_1} j\omega\epsilon\left[\nabla'\psi\cdot(\bar{n}\times\bar{E}_1) + \psi\nabla'\cdot(\bar{n}\times\bar{E}_1)\right] ds \qquad \text{(I)}$$

$$- \frac{1}{4\pi}\int_{s_1}\left[\nabla\psi\cdot\nabla'\times(\bar{n}\times\bar{H}_1) - (\bar{n}\times\bar{H}_1)\cdot\nabla'\times(\nabla\psi)\right] ds \qquad \text{(II), (III)}$$

$$- \frac{1}{4\pi}\int_{s_1}\left[\nabla'(\bar{n}\cdot\bar{H}_1)\cdot\nabla\psi + (\bar{n}\cdot\bar{H}_1)\,\nabla'\cdot(\nabla\psi)\right] ds \qquad \text{(I)}$$

$$\nabla' \cdot \overline{H}'(x', y', z') = -\frac{1}{j\omega\mu}\frac{1}{4\pi}\oint_c \nabla^2 \psi\left(\overline{E}_1 \cdot d\overline{\ell}\right)$$

$$-\frac{1}{4\pi}\int_{s_1}\left[j\omega\epsilon\left(\overline{n}\times\overline{E}_1\right)\cdot\nabla\psi - \left(\overline{n}\cdot\overline{H}_1\right)\nabla^2\psi\right]ds$$

$$= \frac{k^2}{j\omega\mu}\frac{1}{4\pi}\oint_c \psi\overline{E}\cdot d\overline{\ell} - \frac{k^2}{4\pi}\int_{s_1}\left(\overline{n}\cdot\overline{H}_1\right)\psi\, ds$$

$$-\frac{j\omega\epsilon}{4\pi}\int_{s_1}\left(\overline{n}\times\overline{E}_1\right)\cdot\nabla\psi\, ds$$

$$\int_{s_1}\left[\nabla\times\left(\psi\overline{E}_1\right)\right]\cdot\overline{n}\, ds = \oint_c \psi\overline{E}_1\cdot d\overline{\ell} = \int_{s_1}\left(\psi\nabla\times\overline{E}_1 + \nabla\psi\times\overline{E}_1\right)\cdot\overline{n}\, ds \quad \text{(IV), (V)}$$

$$-\left(\nabla\psi\times\overline{E}_1\right)\cdot\overline{n} = \left(\overline{n}\times\overline{E}_1\right)\cdot\nabla\psi \quad \text{(VI)}$$

$$\nabla' \cdot \overline{H}(x', y', z') = \frac{k^2}{j\omega\mu}\frac{1}{4\pi}\oint_c \psi\overline{E}_1\cdot d\overline{\ell} - \frac{k^2}{4\pi}\int_{s_1}\left(\overline{n}\cdot\overline{H}_1\right)\psi\, ds$$

$$+\frac{j\omega\epsilon}{4\pi}\oint_c \psi\overline{E}_1\cdot d\overline{\ell} - \frac{j\omega\epsilon}{4\pi}\int_{s_1}\psi\left(\nabla\times\overline{E}_1\right)\cdot\overline{n}\, ds$$

Then

$$\nabla' \cdot \overline{H}(x', y', z') = 0$$

since

$$\nabla \times \overline{E}_1 = -\frac{\partial}{\partial t}\left(B_1 e^{-j\omega t}\right) = j\omega\mu \overline{H}_1 e^{-j\omega t} .$$

THE CURL OF THE SCATTERED MAGNETIC FIELD

Repeating the general formulation for the scattered magnetic field:

$$\overline{H}(x', y', z') = \frac{1}{j\omega\mu}\frac{1}{4\pi}\oint_c \nabla\psi \, \overline{E}_1 \cdot d\overline{\ell}$$

$$+ \frac{1}{4\pi}\int_s \left[j\omega\epsilon\left(\overline{n} \times \overline{E}_1\right)\psi - \left(\overline{n} \times \overline{H}_1\right) \times \nabla\psi - \left(\overline{n} \cdot \overline{H}_1\right)\nabla\psi\right] ds$$

Maxwell's equation $\nabla \times \overline{H} = \overline{J} + (\partial\overline{D}/\partial t)$ (free space) is satisfied as follows:

$$\nabla' \times \overline{H}(x', y', z') = \frac{1}{j\omega\mu}\frac{1}{4\pi}\oint_c \left[\left(\overline{E}_1 \cdot d\overline{\ell}\right)\nabla' \times \nabla\psi + \nabla'\left(\overline{E}_1 \cdot d\overline{\ell}\right) \times \nabla\psi\right] \qquad \text{(V)}$$

$$+ \frac{1}{4\pi}\int_{s_1} j\omega\epsilon\left[\psi\nabla' \times \left(\overline{n} \times \overline{E}_1\right) + (\nabla'\psi) \times \left(\overline{n} \times \overline{E}_1\right)\right] ds \text{ (V)}$$

$$-\frac{1}{4\pi}\int_{s_1}\left[k^2(\bar{n}\times\bar{H}_1)\psi+(\bar{n}\times\bar{H}_1)\cdot\nabla\nabla\psi\right]ds \quad \textbf{(VII)}$$

$$-\frac{1}{4\pi}\int_{s_1}\left[(\bar{n}\cdot\bar{H}_1)\nabla'\times\nabla\psi+\nabla'(\bar{n}\cdot\bar{H}_1)\times\nabla\psi\right]ds \textbf{(V)}$$

where

$$\nabla'\times\left[(\bar{n}\times\bar{H}_1)\times\nabla\psi\right] = (\bar{n}\times\bar{H}_1)\nabla'\cdot\nabla\psi$$

$$-\left[(\bar{n}\times\bar{H}_1)\cdot\nabla'\right]\nabla\psi-\nabla\psi\nabla'\cdot(\bar{n}\times\bar{H}_1)+(\nabla\psi\cdot\nabla')(\bar{n}\times\bar{H}_1)$$

$$= k^2(\bar{n}\times\bar{H}_1)\psi+(\bar{n}\times\bar{H}_1)\cdot\nabla\nabla\psi . \quad \textbf{(VII)}$$

It can be verified that

$$\int_{s_1}(\bar{n}\times\bar{H}_1)\cdot\nabla\nabla\psi\, ds = \int_{s_1}(\bar{n}\cdot\nabla\times\bar{H}_1)\nabla\psi\, ds - \int_{s_1}\bar{n}\times\nabla\cdot(\bar{H}_1\nabla\psi)\, ds$$

by treating $(\bar{H}_1 \ \nabla\psi)$ as a dyad[1] and writing out the 72 terms of the equality. Derivatives of the illumination distribution with respect to the operator ∇ do not vanish.

[1]Ref. 6, Appendix 3; Ref. 11, Chapters V, VI, and VII; Ref. 12, p. 420, p. 520; Ref. 13, p. 46, p. 113; Ref. 14, See Index.

Using the dyad analogue to Stokes' theorem,

$$\oint_c \nabla\psi(\overline{H}_1 \cdot d\overline{\ell}) = \oint_c (d\overline{\ell} \cdot \overline{H}_1) \nabla\psi = \oint_c d\overline{\ell} \cdot (\overline{H}_1 \nabla\psi)$$

$$= \int_{s_1} \overline{n} \cdot \nabla \times (\overline{H}_1 \nabla\psi) \, ds = \int_s \overline{n} \times \nabla \cdot (\overline{H}_1 \nabla\psi) \, ds \ . \qquad \text{(VIII)}$$

The last equality was verified, term-by-term, by expanding $(\overline{H}_1 \nabla\psi)$, and also follows from identity IX when $\overline{B} \to \nabla$. In general, operators cannot replace vectors or dyads in identity relationships. Therefore,

$$\int_{s_1} (\overline{n} \times \overline{H}_1) \cdot \nabla\nabla\psi \, ds = -j\omega\epsilon \int_{s_1} (\overline{n} \cdot \overline{E}_1) \nabla\psi \, ds - \oint_c \nabla\psi \overline{H}_1 \cdot d\overline{\ell}$$

and

$$\nabla' \times \overline{H}(x', y', z') = \frac{1}{4\pi} \oint_c \nabla\psi \overline{H}_1 \cdot d\overline{\ell}$$

$$+ \frac{j\omega\epsilon}{4\pi} \int_{s_1} \left[(\overline{n} \times \overline{E}_1) \times \nabla\psi + j\omega\mu(\overline{n} \times \overline{H}_1) \psi + (\overline{n} \cdot \overline{E}_1) \nabla\psi \right] ds$$

so that

$$\nabla' \times \overline{H}(x', y', z') = -j\omega\epsilon\,\overline{E}(x', y', z')$$

and Maxwell's first equation is satisfied by the Kirchhoff-Kottler formulation.

THE CURL OF THE SCATTERED ELECTRIC FIELD

Repeating the general formulation for the electric field:

$$\overline{E}(x', y', z') = -\frac{1}{j\omega\epsilon}\frac{1}{4\pi}\oint_c \nabla\psi\,\overline{H}_1 \cdot d\overline{\ell}$$

$$-\frac{1}{4\pi}\int_s \left[j\omega\mu(\overline{n} \times \overline{H}_1)\,\psi + (\overline{n} \times \overline{E}_1) \times \nabla\psi + (\overline{n} \cdot \overline{E}_1)\,\nabla\psi \right] ds$$

Maxwell's equation $\nabla \times \overline{E} = -(\partial\overline{B}/\partial t)$ is satisfied as follows:

$$\nabla' \times E(x', y', z') = -\frac{1}{j\omega\epsilon}\oint_{\partial c} \left[(\overline{H}_1 \cdot d\overline{\ell})\,\nabla' \times \nabla\psi + \nabla'(\overline{H}_1 \cdot d\overline{\ell}) \times \nabla\psi \right] \quad \text{(V)}$$

$$-\frac{1}{4\pi}\int_{s_1} j\omega\mu\left[\psi\nabla' \times (\overline{n} \times \overline{H}_1) + (\nabla'\psi) \times (\overline{n} \times \overline{H}_1)\right] ds \quad \text{(V)}$$

$$- \frac{1}{4\pi} \int_{s_1} \left[k^2 \left(\bar{n} \times \bar{E}_1 \right) \psi + \left(\bar{n} \times \bar{E}_1 \right) \cdot \nabla\nabla\psi \right] ds \qquad \textbf{(VII)}$$

$$- \frac{1}{4\pi} \int_{s_1} \left[\left(\bar{n} \cdot \bar{E}_1 \right) \nabla' \times \nabla\psi + \nabla' \left(\bar{n} \cdot \bar{E}_1 \right) \times \nabla\psi \right] ds \, \textbf{(V)}$$

where

$$\nabla' \times \left[\left(\bar{n} \times \bar{E}_1 \right) \times \nabla\psi \right] = \left(\bar{n} \times \bar{E}_1 \right) \nabla' \cdot \nabla\psi - \left[\left(\bar{n} \times \bar{E}_1 \right) \cdot \nabla' \right] \nabla\psi$$

$$- \nabla\psi \, \nabla' \cdot \left(\bar{n} \times \bar{E}_1 \right) + \left(\nabla\psi \cdot \nabla' \right) \left(\bar{n} \times \bar{E}_1 \right)$$

$$= k^2 \left(\bar{n} \times \bar{E}_1 \right) \psi + \left(\bar{n} \times \bar{E}_1 \right) \cdot \nabla\nabla\psi \qquad \textbf{(VII)}$$

It can be verified that

$$\int_{s_1} \left(\bar{n} \times \bar{E}_1 \right) \cdot \nabla\nabla\psi \, ds = \int_{s_1} \left(\bar{n} \cdot \nabla \times \bar{E}_1 \right) \nabla\psi \, ds - \int_{s_1} \bar{n} \times \nabla \cdot \left(\bar{E}_1 \, \nabla\psi \right) ds$$

by treating $\left(\bar{E}_1 \, \nabla\psi \right)$ as a dyad and writing out the individual terms of the equality.

Using the dyad analogue to Stokes' theorem,

$$\oint_c \nabla\psi(\bar{E}_1 \cdot d\bar{\ell}) = \oint_c (d\bar{\ell} \cdot \bar{E}_1)\,\nabla\psi = \oint_c d\bar{\ell} \cdot (\bar{E}_1\,\nabla\psi)$$

$$= \int_{s_1} \bar{n} \cdot \nabla \times (\bar{E}_1\,\nabla\psi)\,ds = \int_{s_1} \bar{n} \times \nabla \cdot (\bar{E}_1\,\nabla\psi)\,ds. \quad \text{(VIII)}$$

Therefore,

$$\int_{s_1} (\bar{n} \times \bar{E}_1) \cdot \nabla\nabla\psi\,ds = j\omega\mu \int_{s_1} (\bar{n} \cdot \bar{H}_1)\,\nabla\psi\,ds - \oint \nabla\psi\,\bar{E}_1 \cdot d\bar{\ell}$$

and

$$\nabla' \times \bar{E}(x', y', z') = \frac{1}{4\pi} \oint_c \nabla\psi\,\bar{E}_1 \cdot d\bar{\ell}$$

$$+ \frac{j\omega\mu}{4\pi} \int_{s_1} \left[j\omega\epsilon(\bar{n} \times \bar{E}_1) - (\bar{n} \times \bar{H}_1) \times \nabla\psi - (\bar{n} \cdot \bar{H}_1)\,\nabla\psi \right] ds$$

so that

$$\nabla' \times \overline{E}(x', y', z') = j\omega\mu \, \overline{H}(x', y', z')$$

and Maxwell's second equation is satisfied by the Kirchhoff-Kottler formulation.

APPENDIX B

SANCER'S DERIVATIONS (THE FREE-SPACE DYADIC GREEN'S FUNCTION)[1]

Sancer has shown that the complete Kirchhoff-Kottler formulation for $E(x' y' z')$ can be obtained using the free-space dyadic Green's function, thereby avoiding separate physical or intuitive arguments to obtain the contour integral. Sancer has also manipulated the general formulation in such a way as to establish the fact that the radial fields vanish at infinity as required.[2] The special case of a closed surface is treated in terms of two open surfaces using the vector form of Stokes' theorem.[3]

In this appendix, the derivations of Sancer are not reproduced since that author uses conventional notation, operators, etc. throughout. This appendix augments Sancer's article through a discussion of certain dyadic identities, arbitrary vectors, and operators in spherical coordinates. References to texts containing special identities are given, and dyad analogues to expressions known from vector analysis are identified. Wherever the expressions are too involved for complete presentation, the method of obtaining the desired result is simply outlined.

The dyadic Green's function

$$\bar{\bar{G}} = \left(\bar{\bar{I}} + \frac{1}{k^2} \nabla \nabla \right) g$$

where

$$g = \frac{1}{4\pi \left| \bar{r} - \bar{r}' \right|} \exp. \left(ik \left| r - r' \right| \right) ,$$

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[1] Ref. 7, pp. 141-144.

[2] Ref. 2, p. 149 and p. 88. Silver states that "The effect of the boundary-line distribution is therefore to cancel the longitudinal field component introduced by the surface charge and current distributions." Reference is then made to a volume integral argument for the far-zone fields. The Kottler integral is not explicity involved.

[3] Ref. 15, p. 352.

is fundamental to Sancer's development.[1] One of the first relationships involving $\bar{\bar{G}}$ is [2]

$$(L\bar{\bar{G}}) \cdot \bar{V}' = L(\bar{\bar{G}} \cdot \bar{V}')$$

where $\bar{V}'(\bar{r}')$ is an arbitrary vector field, and the operator $L = \nabla \times \nabla \times - k^2$. The given relationship is verified in a straightforward manner by expanding both sides, beginning with the curl of the curl of the dyad $\bar{\bar{G}}$.

Now $\nabla \times \bar{\bar{G}}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \Big(G_{xx}\,\hat{i}\hat{i} + G_{xy}\,\hat{i}\hat{j} + G_{xz}\,\hat{i}\hat{k}$$

$$+ G_{yx}\,\hat{j}\hat{i} + G_{yy}\,\hat{j}\hat{j} + G_{yz}\,\hat{j}\hat{k} + G_{zx}\,\hat{k}\hat{i} + G_{zy}\,\hat{k}\hat{j} + G_{zz}\,\hat{k}\hat{k}\Big)$$

$$= \hat{i}\left(\frac{\partial \bar{G}_z}{\partial y} - \frac{\partial \bar{G}_y}{\partial z}\right) + \hat{j}\left(\frac{\partial \bar{G}_x}{\partial z} - \frac{\partial \bar{G}_z}{\partial x}\right) + \hat{k}\left(\frac{\partial \bar{G}_y}{\partial x} - \frac{\partial \bar{G}_x}{\partial y}\right),$$

where

$$\bar{G}_x = G_{xx}\,\hat{i} + G_{xy}\,\hat{j} + G_{xz}\,\hat{k}$$

$$\bar{G}_y = G_{yx}\,\hat{i} + G_{yy}\,\hat{j} + G_{yz}\,\hat{k}$$

$$\bar{G}_z = G_{zx}\,\hat{i} + G_{zy}\,\hat{j} + G_{zz}\,\hat{k}\,.$$

---

[1] Ref. 6, p. 221 Van Bladel makes extensive use of the dyadic Green's function in solving a large variety of mechanics and electromagnetics problems.

[2] Ref. 7, p. 142, Equation (2.8).

Then $\nabla \times (\nabla \times \bar{\bar{G}})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{i}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[\hat{i}\left(\frac{\partial \bar{G}_z}{\partial y} - \frac{\partial \bar{G}_y}{\partial z}\right)\right.$$

$$\left. + \hat{j}\left(\frac{\partial \bar{G}_x}{\partial z} - \frac{\partial \bar{G}_z}{\partial x}\right) + \hat{k}\left(\frac{\partial \bar{G}_y}{\partial x} - \frac{\partial \bar{G}_x}{\partial y}\right)\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial \bar{G}_y}{\partial x} - \frac{\partial \bar{G}_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \bar{G}_x}{\partial z} - \frac{\partial \bar{G}_z}{\partial x}\right)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial \bar{G}_z}{\partial y} - \frac{\partial \bar{G}_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \bar{G}_y}{\partial x} - \frac{\partial \bar{G}_x}{\partial y}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial \bar{G}_x}{\partial z} - \frac{\partial \bar{G}_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \bar{G}_z}{\partial y} - \frac{\partial \bar{G}_y}{\partial z}\right)\right].$$

The analogy with vector analysis is obvious. In the expression[1] for $\nabla \times \nabla \text{x}\, \bar{V}$, replace vector $\bar{V}$ by the dyad $\bar{\bar{G}}$.

The left-hand side of the equation can now be expanded, and the result dotted into $\bar{V}'$ to form $(L\bar{\bar{G}}) \cdot \bar{V}'$. The right-hand side of the equation is then treated by forming the <u>vector</u>

$$\bar{\bar{G}} \cdot \bar{V}' = \left(G_{xx} V_x' + G_{xy} V_y' + G_{xz} V_z'\right)\hat{i} + \left(G_{yx} V_x'\right.$$

$$\left. + G_{yy} V_y' + G_{yz} V_z'\right)\hat{j} + \left(G_{zx} V_x' + G_{zy} V_y' + G_{zz} V_z'\right)\hat{k}$$

---

[1] Ref. 7, p. 490, identity 59. A correction is required. The last factor should read $\partial^2 v_y/\partial y \partial z$ instead of $\partial^2 v_y/\partial x\, \partial z$.

and writing out $\nabla \times \nabla \times (\bar{\bar{G}} \cdot \bar{V}')$. The identity

$$(L\bar{\bar{G}}) \cdot \bar{V}' = L(\bar{\bar{G}} \cdot \bar{V})$$

is verified by this process, however, it appears that the arbitrary vector field $\bar{V}'(\bar{r}')$ is a <u>constant</u> vector field relative to the unprimed partials. Writing only the $\hat{i}$ component of the equality here, to save space, the left-hand and right-hand sides of the relationship yield

$$V_x'\left(\frac{\partial^2 G_{yx}}{\partial y \partial x} - \frac{\partial^2 G_{xx}}{\partial z^2} - \frac{\partial^2 G_{xx}}{\partial y^2} + \frac{\partial^2 G_{zx}}{\partial z \partial x}\right)$$

$$+ V_y'\left(\frac{\partial^2 G_{yy}}{\partial y \partial x} - \frac{\partial^2 G_{xy}}{\partial z^2} - \frac{\partial^2 G_{xy}}{\partial y^2} + \frac{\partial^2 G_{zy}}{\partial z \partial x}\right)$$

$$+ V_z'\left(\frac{\partial^2 G_{yz}}{\partial y \partial x} - \frac{\partial^2 G_{xz}}{\partial z^2} - \frac{\partial^2 G_{xz}}{\partial y^2} + \frac{\partial^2 G_{zz}}{\partial z \partial x}\right)$$

$$= \frac{\partial^2}{\partial y \partial x}\left(G_{yx} V_x' + G_{yy} V_y' + G_{yz} V_z'\right)$$

$$- \frac{\partial^2}{\partial y^2}\left(G_{xx} V_x' + G_{xy} V_y' + G_{xz} V_z'\right)$$

$$- \frac{\partial^2}{\partial z^2}\left(G_{xx} V_x' + G_{xy} V_y' + G_{xz} V_z'\right)$$

$$+ \frac{\partial^2}{\partial z \partial x}\left(G_{zx} V_x' + G_{zy} V_y' + G_{zz} V_z'\right).$$

It is noted that the dot product of a dyad with a vector is non-commutative in general. The curl of the curl of a dyad is a dyad, and a dyad dotted with a vector is a vector.

After applying the divergence theorem of Gauss,[1] Sancer writes

$$\bar{\bar{G}} \cdot \bar{V}' = \bar{V}' \cdot \bar{\bar{G}} ,$$

which expresses the fact that $\bar{\bar{G}}$ is symmetric. As expected, an expansion of the equality shows that the symmetry refers to the principal diagonal symmetry of the matrix representation of $\bar{\bar{G}}$, so that $G_{ij} = G_{ji}$ when the i, j are the row-column designation. Sancer next employs a relationship

$$\nabla \times (\bar{\bar{G}} \cdot \bar{V}') = \nabla g \times \bar{V}'$$

which is easily verfied by expanding the left-hand side and writing out the expression $\bar{\bar{G}} = (\bar{\bar{I}} + 1/k^2 \nabla\nabla) g$. Since

$$\nabla\nabla = \hat{i}\hat{i} \frac{\partial^2}{\partial x^2} + \hat{i}\hat{j} \frac{\partial^2}{\partial x \partial y} + \hat{i}\hat{k} \frac{\partial^2}{\partial x \partial z}$$

$$+ \hat{j}\hat{i} \frac{\partial^2}{\partial y \partial x} + \hat{j}\hat{j} \frac{\partial^2}{\partial y^2} + \hat{j}\hat{k} \frac{\partial^2}{\partial y \partial z} + \hat{k}\hat{i} \frac{\partial^2}{\partial z \partial x} + \hat{k}\hat{j} \frac{\partial^2}{\partial z \partial y} + \hat{k}\hat{k} \frac{\partial^2}{\partial z^2} ,$$

$$\nabla \times (\bar{\bar{G}} \cdot \bar{V}') = \hat{i} \left[\frac{\partial}{\partial y} (gV_z) - \frac{\partial}{\partial z} (gV_y)\right]$$

$$+ \hat{j} \left[\frac{\partial}{\partial z} (gV_x) - \frac{\partial}{\partial x} (gV_z)\right] + \hat{k} \left[\frac{\partial}{\partial x} (gV_y) - \frac{\partial}{\partial y} (gV_x)\right] .$$

[1]See the identities X and XI in Appendix A.

Then the identity is verified when $\overline{V}'$ is an arbitrary constant vector field relative to the unprimed partials.

After a considerable amount of manipulation,[1] which is straightforward, but tedious to verify, Sancer arrives at a result[2] that is written in his notation as

$$\overline{E}(\overline{r}) = \overline{E}_i(\overline{r}) + \int_{s'} \Big\{ \underbrace{(\overline{n}' \times \overline{E}') \times \nabla' g}_{3}$$

$$+ \underbrace{i\omega\mu g\, (\overline{n}' \times \overline{H}')}_{2} + \frac{i}{\omega\epsilon} \nabla'\nabla' g \cdot (\overline{n} \times \overline{H}) \Big\}\, ds\ .$$

The integrals (2) and (3) of Stratton's formulation can be identified. Further manipulation which employs various artifices such as adding and subtracting the same integral, finally results in[3]

$$\overline{E}(\overline{r}) = \overline{E}_i(\overline{r}) + \int_{S_I'} \Big\{ \underbrace{(\overline{n} \times \overline{E}_k') \times \nabla' g}_{3}$$

$$+ \underbrace{i\omega\mu g \left(n' \times \overline{H}_k'\right)}_{2} + \underbrace{\nabla' g \left(\overline{n} \cdot \overline{E}_k'\right)}_{4} \Big\}\, ds_I'$$

$$- \underbrace{\oint \frac{i}{\omega\epsilon} \nabla' g\, \overline{H}_k' \cdot d\overline{\ell}'}_{1}$$

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[1] The triple scalar product and identity IX of Appendix B are used.
[2] Ref. 7, p. 142, Equation (2.17).
[3] Ref. 7, p. 143, Equation (2.24).

which can be identified as the original formulation of Stratton.[1] The existence of the contour integral is especially significant since it was obtained directly via the free-space dyadic Green's function. The intermediate expression, above, will be used subsequently to investigate the radial fields in the far-zone.

Returning to an argument by Sancer[2] concerning the contour integral for closed surfaces,

$$\overline{\nabla}' \cdot \oint \frac{i}{\omega\epsilon} \nabla g \left(\overline{H}_1 - \overline{H}_2\right) d\overline{\ell} \ .$$

is said to vanish when $\overline{H}$ is continuous on the surface due to Stokes' theorem for a closed surface. The argument is subtle since, if $\overline{H}$ is continuous, $\overline{H} = \overline{H}_1 = \overline{H}_2 = 0$ and it is easy to lose sight of the fact that it is the contour which vanished.[3]

The following alternative viewpoint is presented to avoid the a priori statement that the surface integral vanishes, since it is the vanishing of the contour integral which establishes the far-less obvious vanishing of the surface integral. Obviously, the presence of the arbitiary field vector $\overline{V}'$ allows Sancer to employ the vector form of Stokes' theorem. The behavior of the contour integral can be determined without the use of $\overline{V}'$ by using the dyad analogue to Stokes' theorem.

From Stokes, in Stratton's notation,

$$\oint_{c_1} \nabla\psi \overline{H}_1 \cdot d\overline{\ell} = \oint_{c_1} d\overline{\ell} \cdot (\overline{H}_1 \nabla\psi) = \int_{s_1} \overline{n} \cdot \nabla x (\overline{H}_1 \nabla\psi)\, ds \ ,$$

visualizing a simple open surface s_1 initially. The contour is by common convention, oriented so that the right-hand rule holds for the normal to the surface. If a second simple open surface should exist such that the boundary c_1 is also

[1]Ref. 1, p. 469.
[2]Ref. 7, p. 143, Equation (2.23).
[3]Ref. 15, p. 352.

the boundary $-c_2$, it follows that

$$\int_{s_1+s_2=s} \overline{n} \cdot \nabla \times \left(\overline{H}_1\, \nabla\psi\right) ds \;=\; \oint_c d\overline{\ell} \cdot \left(\overline{H}_1\, \nabla\psi\right)$$

$$= \oint_{c_1} d\overline{\ell}_1 \cdot \left(\overline{H}_1\, \nabla\psi\right) + \oint_{c_2} d\overline{\ell}_2 \cdot \left(\overline{H}_1\, \nabla\psi\right) = \oint_c \left(d\overline{\ell}_1 + d\overline{\ell}_2\right)\left(\overline{H}_1\, \nabla\psi\right) = 0$$

because the surfaces and contours are orientable. That is $d\overline{\ell}_1 = -d\overline{\ell}_2$. Then the conclusion is that the integral over the closed surface also vanishes,[1] necessarily, since the contour has been annihilated.

Sancer's intermediate result

$$\overline{E}(\overline{r}) \;=\; \overline{E}_i(\overline{r}) + \int_s \left\{ (\overline{n}' \times \overline{E}') \times \nabla' g \right.$$

3

$$\left. + \; i\omega\mu g\, (\overline{n}' \times \overline{H}') \;+\; \frac{i}{\omega\epsilon} \nabla'\nabla' g \cdot (\overline{n} \times \overline{H}') \right\} ds$$

2

leads to the conclusion that the Kirchhoff-Kottler formulation exhibits transverse fields only in the far-zone limit. Recalling the dyadic Green's function,

$$\overline{\overline{G}} \;=\; \left(\overline{\overline{I}} + \frac{1}{k^2} \nabla\nabla \right) g \;.$$

[1] Ref. 16, p. 338. Several examples of orientable and non-orientable surfaces are given with sketches, and direct reference to Stokes' theorem is also made when $S = S_1 + S_2$ as above.

Sancer writes

$$\bar{E}(\bar{r}) = \bar{E}_i(\bar{r}) + \frac{i\omega\mu e^{ikr}}{4\pi r}\left(\bar{\bar{I}} - \bar{a}_r\bar{a}_r\right)$$

$$\cdot \int_{s'} (\bar{n} \times \bar{H}')e^{-ik\bar{a}_r \cdot \bar{r}'}\, ds' + \frac{ike^{ikr}}{4\pi r}\bar{a}_r \times \int_{s'} (\bar{n}' \times \bar{E}')e^{-ik\bar{a}_r \cdot \bar{r}'}\, ds' ,$$

after injecting the far-field approximation $e^{-ik\bar{a}_r \cdot r'}$, and rewriting $\bar{\bar{G}}$ in spherical coordinates in the limit as $r \to \infty$.

The $\bar{\bar{G}}^{\infty}(r)$ follows from the spherical operator

$$\nabla = \hat{i}_r \frac{\partial}{\partial r} + \hat{i}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{i}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} ,$$

and the basis vectors of the moving spherical triad

$$\hat{i}_r = \sin\theta\cos\phi\ \hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\,\hat{k}$$

$$\hat{i}_\theta = \cos\theta\cos\phi\ \hat{i} + \cos\theta\sin\phi\ \hat{j} - \sin\theta\,\hat{k}$$

$$\hat{i}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Then

$$\nabla\nabla g = \hat{\imath}_r \frac{\partial}{\partial r}\left(\hat{\imath}_r \frac{\partial g}{\partial r}\right) + \hat{\imath}_r \frac{\partial}{\partial r}\left(\hat{\imath}_\theta \frac{1}{r}\frac{\partial g}{\partial \theta}\right) + \hat{\imath}_r \frac{\partial}{\partial r}\left(\hat{\imath}_\phi \frac{1}{r\sin\theta}\frac{\partial g}{\partial \phi}\right)$$

$$+ \frac{\hat{\imath}_\theta}{r}\frac{\partial}{\partial \theta}\left(\hat{\imath}_r \frac{\partial g}{\partial r}\right) + \frac{\hat{\imath}_\theta}{r}\frac{\partial}{\partial \theta}\left(\hat{\imath}_\theta \frac{1}{r}\frac{\partial g}{\partial \theta}\right) + \frac{\hat{\imath}_\theta}{r}\frac{\partial}{\partial \theta}\left(\hat{\imath}_\phi \frac{1}{r\sin\theta}\frac{\partial g}{\partial \phi}\right)$$

$$+ \frac{\hat{\imath}_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\hat{\imath}_r \frac{\partial g}{\partial r}\right) + \frac{\hat{\imath}_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\hat{\imath}_\theta \frac{1}{r}\frac{\partial g}{\partial \theta}\right) + \frac{\hat{\imath}_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\hat{\imath}_\phi \frac{1}{r\sin\theta}\frac{\partial g}{\partial \phi}\right)$$

$$= \hat{\imath}_r \hat{\imath}_r \frac{\partial^2 g}{\partial r^2} + \frac{\hat{\imath}_\theta \hat{\imath}_\theta}{r}\frac{\partial g}{\partial r} + \frac{\hat{\imath}_\phi \hat{\imath}_\phi}{r}\frac{\partial g}{\partial r}\,.$$

$$= \hat{\imath}_r \hat{\imath}_r \left(\frac{2}{r^2} - \frac{2ik}{r} - k^2\right) g + \hat{\imath}_\theta \hat{\imath}_\theta \left(\frac{ik}{r} - \frac{1}{r^2}\right) g + \hat{\imath}_\phi \hat{\imath}_\phi \left(\frac{ik}{r} - \frac{1}{r^2}\right) g$$

From this representation of $\nabla\nabla g$,[1]

$$\bar{\bar{G}}(r) = \left(\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla\right) g$$

$$= \left(\bar{\bar{I}} - \bar{a}_r \bar{a}_r\right) g + \left(\frac{i}{kr} - \frac{1}{k^2 r^2}\right)\left(\bar{\bar{I}} - 3\bar{a}_r \bar{a}_r\right) g\,.$$

[1]The various authors use different symbols to represent the unit radial vector. Here $\hat{\imath}_r = \bar{a}_r$ (Sancer) $= a_r$ (Van Bladel) $= \hat{r}$ (Newton) which is usually clear from the context. Also, $g = e^{ikr}/r$ (Sancer) and $g = e^{-jkr}/r$ (Van Bladel).

In the limit[1] as $r \to \infty$,

$$\overline{\overline{G}}^{\infty}(\overline{r}) = (\overline{\overline{I}} - \overline{a}_r \overline{a}_r) g = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} g$$

as used by Sancer. When $\overline{\overline{G}}^{\infty}(r)$ operates in a dot product against an integral of the form

$$\int_s (\overline{n} \times \overline{H}') e^{-ika_r \cdot r'} \, dS'$$

the result can be regarded as the entire vector field minus the radial part of that field via $(\overline{\overline{I}} - \overline{a}_r \overline{a}_r) g$. Alternatively, the result can be regarded as the dot product of a dyad $\overline{\overline{G}}^{\infty}(r)$ which has only $\hat{1}_\theta \hat{1}_\theta$ and $\hat{1}_\phi \hat{1}_\phi$ components in a matrix, with a field of $\hat{1}_r$, $\hat{1}_\theta'$, $\hat{1}_\phi$ values. This is equivalent to taking only the transverse components of the field.

To sum up, Sancer has shown that the Kirchhoff-Kottler formulation has only transverse fields in the far-zone limit since (1)

$$\int_{s'} (\overline{n}' \times \overline{E}') \, x' \nabla g \, ds$$

is necessarily transverse (but vanishes anyway when the conductor is perfectly reflecting), and (2) the radial part of the integral for electric sheet current is annihilated by an integral which is equivalent to the boundary curve integral on electric charge plus the integral on surface-charge distribution.

[1]Ref. 7, p. 221 and Ref. 17, p. 1??